Notes

Nonlinear Pressure Vessel Stress Analysis Using the Optimum Programming Approach

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Optimum Programming Technique

THE optimum programming technique was first developed by Bliss.[1] Typical uses of the technique are found in Refs. 2 and 3. In the optimum programming technique, systems behavior is defined through a set of nonlinear ordinary differential equations of the first order. The independent variable may be either time or a characteristic dimensional quantity. In the system of equations, there may be one or more parameters that are controllable and are thus subject to adjustment. In other cases, the parameters are of unknown magnitudes because of ignorance. The guidance and control problem of a missile belongs to the former category, whereas the shell stress problem belongs to the latter category. The purpose of optimum programming is to find a best set of values of the parameters or control variables, so as to optimize the terminal value of a given function, to be called the optimized function. At the same time, the control variables should be adjusted in such a manner that certain boundary conditions are satisfied.

In the optimum programming technique, a set of nominal or most reasonable values of the control variables is first assumed. Based on the nominal values of the control variables, the systems equations are solved numerically. A set of "adjoint functions" is established which measures the influence of the change of a control variable at a specific station to the corresponding changes of the optimized function and the unsettled boundary conditions at the terminal point. It is possible to adjust the control variables in small steps so as to reduce the optimized function gradually to its optimum value and to make the state variables satisfy certain boundary conditions at the terminal point.

Problem Formulation

Briefly, the optimum program technique is to determine the control variables $\alpha(s)$ and to solve for $x(s)$ based on the following set of differential equations:

$$dx/ds = f(x,\alpha,s) \tag{1}$$

in the given interval $s_0 \leqslant s \leqslant S$. In Eq. (1), $x(s)$ is an $(n \times 1)$ matrix representing the state variables:

$$x(s) = \begin{bmatrix} x_1(s) \\ \cdot \\ \cdot \\ x_n(s) \end{bmatrix}$$

$\alpha(s)$ is an $(m \times 1)$ matrix representing the control variables, which are subject to adjustment in the optimization process:

$$\alpha(s) = \begin{bmatrix} \alpha_1(s) \\ \cdot \\ \cdot \\ \alpha_m(s) \end{bmatrix}$$

f is an $(n \times 1)$ matrix representing the derivatives (dx/ds), which are known functions of $x(s)$, $\alpha(s)$ and the independent variable s:

$$f = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{bmatrix}$$

The initial values of the state variables $x(s_0)$ are given in certain problems. In the case of the nonlinear shell stress problems, only part of the initial values $x(s_0)$ are given. The purpose of optimum programming is to determine $\alpha(s)$ in the given s-interval as well as the initial values of the state variables $x(s_0)$ so as to optimize the function $\Phi[x(S),S]$ and to satisfy the terminal boundary conditions:

$$\psi[x(S),S] = 0 \tag{2}$$

Starting with a set of nominal values of $\alpha(s)$ and $x(s_0)$, the technique postulates the following small amplitude corrections for $\alpha(s)$ and the undetermined initial conditions $x(s_0)$ for given δP:

$$\delta\alpha = (1/2\mu)W^{-1}(\partial f/\partial\alpha)'(\lambda_\phi - \lambda_\psi\nu_1) \tag{3}$$

$$\delta x(s_0) = (1/2\mu)Y^{-1}[\lambda_\phi(s_0) - \lambda_\psi(s_0)\nu_1] \tag{4}$$

In Eqs. (3) and (4), $\lambda_\phi,\lambda_\psi$ are functions adjoint to $x(s)$, and W,Y are weighting functions; ν_1 and μ are defined as

$$\nu_1 = -2\mu\, I_{\psi\psi}{}^{-1}d\psi + I_{\psi\psi}{}^{-1}I_{\psi\phi} \tag{5}$$

$$2\mu = \pm\left[\frac{I_{\phi\phi} - I_{\psi\phi}{}'I_{\psi\psi}{}^{-1}I_{\psi\phi}}{(\delta P)^2 - d\psi' I_\psi{}^{-1}d\psi}\right]^{1/2} \tag{6}$$

where

$$I_{\psi\psi} = \int_{s_0}^{S} \lambda_\psi{}'\left(\frac{\partial f}{\partial\alpha}\right)W^{-1}\left(\frac{\partial f}{\partial\alpha}\right)'\lambda_\psi ds + \lambda_\psi{}'(s_0)Y^{-1}\lambda_\psi(s_0) \tag{7}$$

$I_{\psi\phi},I_{\phi\phi}$ are obtained by replacing either one or both subscripts ψ by ϕ in each term on the right-hand side of Eq. (7). Corrections (3) and (4) are made repeatedly until the conditions of optimization are satisfied. For a detailed description of the technique, the reader is referred to Ref. 3.

Axisymmetrical Shell

Consider an axisymmetrical shell. The deformation and stress patterns, as well as the external loads, are indicated in Fig. 1. In the classical approach, the equilibrium equations are established which are solved together with the stress-strain relations and the compatibility equations.[4] For nonlinear shells where the sectional properties are depending

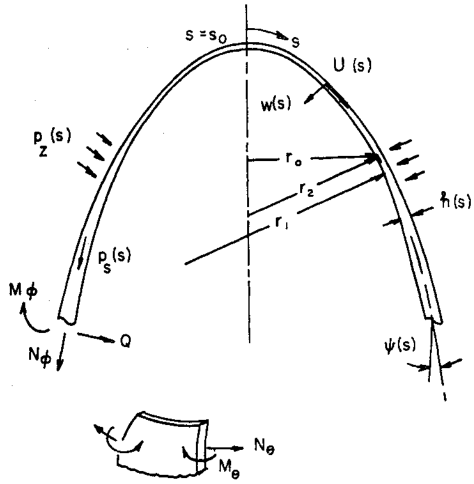

Fig. 1 An axisymmetrical shell.

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Table 1 Terminal values of the adjoint functions

	λ_u	λ_w	λ_ψ	λ_ϕ
For Φ	0	0	0	1
For ψ_1	0	0	1	0
For ψ_2	0	1	0	0
For ψ_3	1	0	0	0

on the curvilinear coordinate s, or in the case of large deflections, the relations between the shell deformations and strains are further complicated.

In the optimum programming approach, certain strain variables in the shell are selected as the control variables α. Additional strain and deformation variables are chosen as the state variables x. The stresses (forces and moments) in the shell are presented in terms of the strains. A nominal condition is established by assuming reasonable values along the shell for the control variables and the undetermined initial values of the state variables. An additional state variable is introduced which is the potential energy in the shell under the nominal condition. This state variable serves in the same time as the optimized function Φ. The stress and strain patterns in the shell which satisfy all the boundary conditions and which correspond to an optimized Φ, i.e., a minimum potential energy, are the solution of the shell stress problem. The object is then to find a best combination of the shell strains that serve as the control variables while the condition of optimization is satisfied. In this manner, the shell stress problem is reduced into a problem of optimum programming.

Before establishing the differential equations for the shell stress problem, the following expressions for the shell strains and stresses are introduced:

$$\psi = \frac{u}{r_1} + \frac{dw}{ds} \tag{8}$$

$$e_\phi = \frac{du}{ds} - \frac{w}{r_1} + \frac{1}{2}\left[\left(\frac{du}{ds} - \frac{w}{r_1}\right)^2 + \left(\frac{u}{r_1} + \frac{dw}{ds}\right)^2\right] \tag{9}$$

$$e_\theta = \frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2} + \frac{1}{2}\left(\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2}\right)^2 \tag{10}$$

$$k_\phi = \frac{d}{ds}\left(\frac{u}{r_1} + \frac{dw}{ds}\right) = \frac{d\psi}{ds} \tag{11}$$

$$k_\theta = \frac{dr_0}{ds}\frac{\psi}{r_0} \tag{12}$$

$$N_\phi = \frac{Eh}{1 - \nu^2}(e_\phi + \nu e_\theta) \tag{13}$$

$$N_\theta = \frac{Eh}{1 - \nu^2}(e_\theta + \nu e_\phi) \tag{14}$$

$$M_\phi = -D(k_\phi + \nu k_\theta) \tag{15}$$

$$M_\theta = -D(k_\theta + \nu k_\phi) \tag{16}$$

In expressions (8–16), the majority of the variables are defined in Fig. 1. In addition, e_ϕ, e_θ, k_ϕ, k_θ are the normal and flexural strains; E, ν, and D are Young's modulus, Poisson's ratio, and the shell flexural rigidity, respectively.

Shell Stress Problem in Optimum Programming Format

In the shell stress problem, e_ϕ and k_ϕ are selected as the control variables. Making use of expressions (8–16), we have the following equations for the axisymmetrical shell:

$$\frac{du}{ds} = e_\phi + \frac{w}{r_1} - \frac{1}{2}(e_\phi{}^2 + \psi^2) \tag{17}$$

$$\frac{dw}{ds} = \psi - \frac{u}{r_1} \tag{18}$$

$$\frac{d\psi}{ds} = k_\phi \tag{19}$$

$$\begin{aligned}\frac{d\Phi}{ds} = 2\pi r_0\Bigg\{&\frac{1}{2}\frac{Eh}{1-\nu^2}\left[\!\left[e_\phi + \nu\left[\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2} + \frac{1}{2}\left(\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2}\right)^2\right]\right]\!\right] e_\phi + \frac{1}{2}\frac{Eh}{1-\nu^2}\left[\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2} + \frac{1}{2}\times\right.\\ &\left.\left(\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2}\right)^2 + \nu e_\phi\right] \times \left[\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2} + \frac{1}{2}\left(\frac{dr_0}{ds}\frac{u}{r_0} - \frac{w}{r_2}\right)^2\right] + \\ &\frac{1}{2}D\left(k_\phi + \nu\frac{dr_0}{ds}\frac{\psi}{r_0}\right)k_\phi + \frac{1}{2}D\left(\frac{dr_0}{ds}\frac{\psi}{r_0} + \nu k_\phi\right)\frac{dr_0}{ds}\frac{\psi}{r_0} - \\ &p_s u - p_z w\Bigg\}\end{aligned} \tag{20}$$

where Φ is the potential energy in the shell structure and is the optimized function. Among the four state variables u, w, ψ, Φ, the initial value $w(s_0)$ is unknown, whereas the other three state variables vanish at $s = s_0$ for a shell without a pointed apex. The boundary conditions at the terminal point of the shell $s = S$ vary for each individual case.

Example

The application of the optimum programming technique is best illustrated by an example. As the technique is essentially a numerical method, it is immaterial whether the stress problem is linear or nonlinear. We choose as an example the stress problem of a conical shell of constant thickness, because the stress data based on established method are available for comparison.

For a conical shell that has a built-in edge at the base $s = l$, its generatrix makes an angle β with the axis of symmetry. It is under constant external pressure p_z. If the shell extension due to the large deflection effect is ignored, Eqs. (17–20) are reduced to the following:

$$du/ds = e_\phi \tag{21}$$

$$dw/ds = \psi \tag{22}$$

$$d\psi/ds = k_\phi \tag{23}$$

$$\begin{aligned}\frac{d\Phi}{ds} = 2\pi s\sin\beta\Bigg\{&\frac{1}{2}\frac{Eh}{1-\nu^2}\left[e_\phi{}^2 + \nu e_\phi\left(\frac{u - w\cot\beta}{s}\right)\right] + \\ &\frac{1}{2}\frac{Eh}{1-\nu^2}\left[\left(\frac{u - w\cot\beta}{s}\right)^2 + \nu e_\phi\left(\frac{u - w\cot\beta}{s}\right)\right] + \\ &\frac{1}{2}D\left[k_\phi{}^2 + \nu k_\phi\frac{\psi}{S}\right] + \frac{1}{2}D\left[\left(\frac{\psi}{s}\right)^2 + \nu k_\phi\frac{\psi}{s}\right] - p_z w\Bigg\}\end{aligned} \tag{24}$$

$$\left.\begin{aligned}&\text{At } s = 0,\ u(0) = w(0)\cot\beta = \text{unknown},\\ &\qquad\qquad \psi(0) = \Phi(0) = 0\\ &\text{At } s = l,\ \psi(l) = w(l) = u(l) = 0\\ &\text{Optimize } \Phi(l)\end{aligned}\right\} \tag{25}$$

Thus, we have the three terminal boundary conditions (25), which are the equivalent of Eq. (2). The adjoint differential equations are

$$\frac{d\lambda_u}{ds} = -\pi\sin\beta\left\{\frac{Eh}{1-\nu^2}\nu e_\phi + \frac{Eh}{1-\nu^2}\left[2\left(\frac{u - w\cot\beta}{s}\right) + \nu e_\phi\right]\right\}\lambda_\phi \tag{26}$$

$$\frac{d\lambda_w}{ds} = \pi\sin\beta\left\{\frac{Eh}{1-\nu^2}\nu\cot\beta + \frac{Eh}{1-\nu^2}\left[2\cot\beta\left(\frac{u - w\cot\beta}{s}\right) + \nu\cot\beta e_\phi\right] + 2sp_z\right\}\lambda_\phi \tag{27}$$

$$d\lambda_\psi/ds = -\lambda_w - \pi\sin\beta\{D\nu k_\phi + D[2(\psi/s) + \nu k_\phi]\}\lambda_\phi \tag{28}$$

$$d\lambda_\phi/ds = 0 \tag{29}$$

At $s = l$, we have the boundary conditions listed in Table 1.

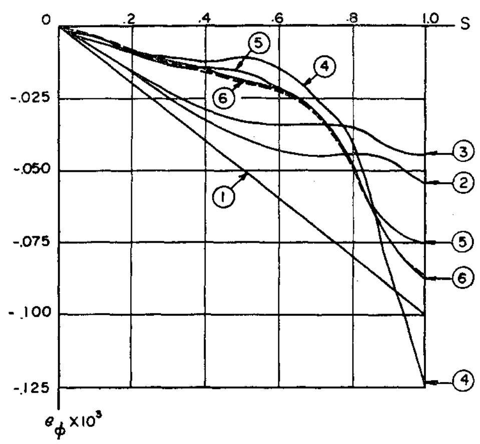

Fig. 2a The longitudinal strain used as the control variable in the stress analysis of a conical shell. The encircled numbers indicate the sequence of iterations performed. The dotted line indicates the solution based on the membrane theory and the homogeneous solution.

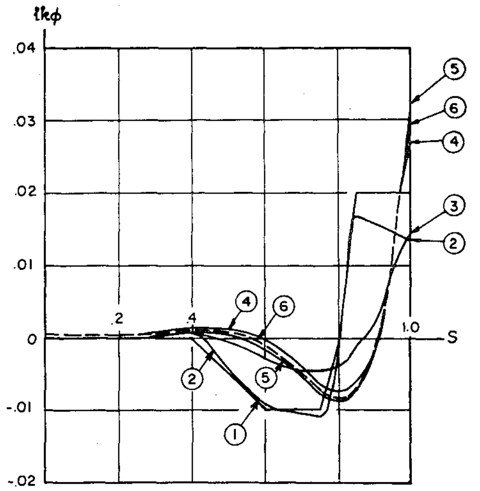

Fig. 2b The longitudinal curvature change used as the control variable in the stress analysis of a conical shell. The encircled numbers indicate the sequence of iterations performed. The dotted line indicates the solution based on the membrane theory and the homogeneous solution.

For the two control variables e_ϕ, k_ϕ, we assume the starting values throughout the range $s = (0,l)$ as shown by curves marked (1) in Fig. 2. The displacements u, w at $s = 0$ are assumed to be zero. We further adapt the following data in the computation: $l/h = 53.034$, $E = 30 \times 10^6$ psi, $p_z = 80$ psi, $h = \frac{1}{2}$ in., $\nu = \frac{1}{3}$, and $\beta = 60°$. The resulting control variable data during various steps of optimization are plotted in Fig. 2. Also plotted are the same data based on the membrane theory and the homogeneous solution. Based on the control variable data and the corresponding values of u, w, and ψ given by Eqs. (21–23), the stress data $N_\phi, N_\theta, M_\phi, M_\theta$ can be obtained from Eqs. (9–16).

Conclusion

In the optimum programming process, the control variables are selected in such a manner that the system can be defined completely with a set of first-order nonlinear differential equations. The control variables appear in both the derivatives of the state variables and the optimized function. In the shell stress problem, the strain variables are chosen as the control variables such that the deformation pattern of the shell is defined. An arbitrarily selected deformation pattern yields higher potential energy in the shell structure under the given load condition as compared to the actual case. Gradual optimization of the potential energy in the shell structure leads to the actual strain and deformation patterns. As the method is a numerical one, many nonlinear shell stress problems not subject to classical analysis can be handled in this manner.

During the optimization process, some judgment is needed in determining the square root mean (δP) of the amplitude change in control variables and the weighting functions W, Y. In general, (δP) should be reduced gradually when the optimization process approaches its final solution. The weighting functions should be arranged in such a way that the more sensitive areas have a higher weighting value.

In the numerical example, the obtained results based on optimum programming compare favorably with the available data based on the membrane and homogeneous solutions.

References

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Mode Shape Effects on Winged Booster Stability

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As future boosters become larger and more flexible, their elastic normal mode frequencies will decrease and narrow the margin between the first elastic mode frequency and the rigid body short-period frequency. When these frequencies become of the same order of magnitude, a coupling phenomenon which may result in an instability in either or both modes can occur. The coupling is particularly severe when the booster has a lifting payload and/or stabilizing fins.

The purpose of this note is to indicate the sensitivity of this coupling to small variations in elastic mode shape. If it is highly sensitive, then the elastic modes must be determined very accurately in order to ensure an accurate determination of the vehicle's forward-loop dynamic stability characteristics. There is somewhat of a paradox here in that the flexible modes become more difficult to determine, either analytically or experimentally, as the boosters become larger. This is due, in large part, to relative motion between structural elements.

When this coupling is very sensitive to mode shape variations, or what is analogous, errors in mode shape determination, the task of designing a suitable control system becomes extremely difficult. In order to maintain transient aerodynamic loading within prescribed limits, it is necessary to provide means of maintaining the desired degree of damping of transient rigid-body oscillations about a nominal trajectory. This must be done primarily by the control system. If small elastic mode shape variations cause wide excursions in short-period damping, the actuation devices would have to be capable of higher frequency response.

The design of a control system with minimum compensation demands an accurate knowledge of the elastic mode shapes and frequencies. As uncertainty in the knowledge of

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